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LETTER TO THE EDITOR

Level-spacing distributions in quasiperiodic systems with metal–insulator transitions

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Abstract. Energy-level statistics in a quasiperiodic system possessing a metal–insulator transition is investigated. For this purpose, the integrated level-spacing distribution $I(s)$ for the Harper model with an incommensurate potential $u_i = \lambda\cos(2\pi\sigma i)$ is numerically calculated. It is shown that there are only three possible distributions. In the metallic regime $\lambda < 2$, $I(s)$ is of the form $(2/\pi)\cos^{-1}(2As/\pi)$, where A is a constant. In the insulating regime $\lambda > 2$, $I(s)$ follows the same law as it does in the situation where the energy levels are equal to u_i. At the critical point $\lambda = 2$, $I(s)$ is described by an inverse power law $I(s) \sim s^{-\beta}$ with $\beta = 1/2$. The derivative of $I(s)$ indicates that level-spacing distributions for the Harper model in the metallic and insulator regimes are different from the well known Wigner surmise and Poisson law for disordered systems.

Level statistics is an important tool in the study of complicated objects such as large atoms, mesoscopic solids, and quantum systems with chaotic behaviour in the classical limit [1–3]. This technique was recently applied to deal with the electronic localization problem in disordered systems [4–6]. It was shown that the metal–insulator transition in the three-dimensional Anderson model can be described by the level-spacing distribution (LSD) $P(s)$, which is defined as the probability density of level spacings s of the adjacent levels. In the metallic regime, $P(s)$ follows the Wigner surmise $P_W(s) \sim s\exp(-cs^2)$, where c is a constant. On the insulating side, the spacings are distributed according to the Poisson law $P_P(s) \sim \exp(-s)$. Thus, as disorder increases, the metal–insulating transition is accompanied by a transition of the spacing distributions from the Wigner surmise to the Poisson law. In a one-dimensional disordered system with the Anderson Hamiltonian, electronic states are localized. It can be proved that the LSD of the spectrum also follows the Poisson law [6]. The Wigner surmise demonstrates level repulsion between energy levels for $P_W(s) \to 0$ when $s \to 0$. On the other hand, since $P_P(s) \neq 0$ when $s \to 0$, level clustering occurs if the distribution follows the Poisson law.

The introduction of quasiperiodicity into solid state physics has greatly enriched the study of electronic localization [7–9]. Interest arises from not only the pure theoretical aspects but also experiments, e.g., the discovery of quasicrystals and the successful preparation of quasiperiodic semiconductor superlattices. Level statistics in the quasiperiodic system of the Harper model was investigated [10–12]; this has the tight-binding Hamiltonian $H = \sum_i u_i|i\rangle\langle i| + \sum_{i,j} v|i\rangle\langle j|$, where $u_i = \lambda\cos(2\pi\sigma i)$ with $\sigma = (\sqrt{5}-1)/2$ is the site energy at the ith site. In the second sum of H, only interactions with the nearest neighbours are considered and $v = 1$. In addition to its quasiperiodic

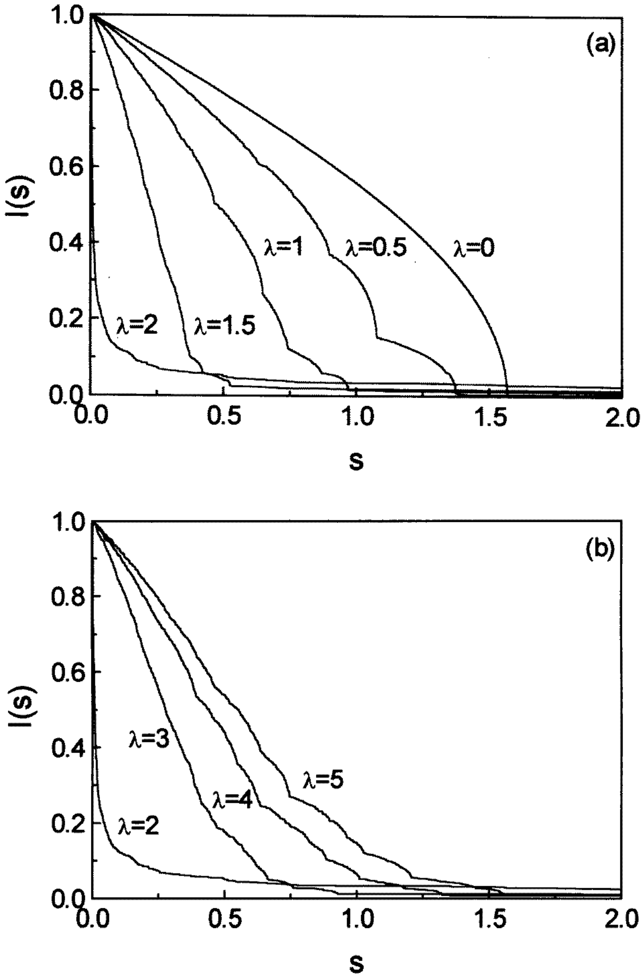

Figure 1. ILSDs $I(s)$ for the Harper model with $N = 987$ and various modulations λ.

nature, the Harper model is directly related to the Bloch electrons in a magnetic field [13]. This model is also relevant to the study of quantum systems with chaotic behaviour in the classic limit [17]. The Harper model displays a metal–insulator transition similar to that of the Anderson Hamiltonian in disordered systems [8, 9, 14]. In the regime $\lambda < 2$, the spectrum is absolutely continuous and eigenstates are extended. For $\lambda > 2$, the spectrum is pointlike and eigenstates are localized. $\lambda = 2$ is the critical point at which the spectrum is a Cantor-like set and the state is critical.

Machida and Fujita [10] found three distinctive distributions for the Harper model: the Poisson type for $\lambda > 2$, an inverse power law at $\lambda = 2$, and a complected distribution for $\lambda < 2$ (they failed to find a single-function form to fit such a distribution). The inverse power law at $\lambda = 2$ was further confirmed by Geisel *et al* [12]. Megann and Ziman [11] studied the level statistics of the Harper model in a different way by introducing new spacings normalized by the average local density of states. However, their calculation does not provide any information in the comparison of the level statistics between the Harper model and disordered systems. In this letter, we study the level statistics for the Harper model. Particular attention will be paid to the shape analysis of the LSD in the metallic and insulating regimes to examine the difference of the level statistics between quasiperiodic systems and disordered structures. We will show that spacing distributions in the regimes $\lambda < 2$ and $\lambda > 2$ follow two simple laws which are different from the distributions suggested by Machida and Fujita [10].

We numerically treat the finite-size cluster with number of sites $N = F_l$ (F_l is the Fibonacci number which is associated with the golden mean $\sigma = \lim_{l\to\infty} F_{l-1}/F_l$). The energy levels are calculated by the direct diagonalization of an $N \times N$ Hamiltonian matrix. For the quasiperiodic system, the density of states has huge fluctuations at all energy scales. In this case, one calculates the level-spacing distribution without any unfolding [18, 10, 12] and defines the level spacing s as $s = (E_{i+1} - E_i)/(W/N)$, where $W = E_{\max} - E_{\min}$. We will look to the integrated level-spacing distribution (ILSD) $I(s)$ defined as $I(s) = \int_s^\infty P(s')\,\mathrm{d}s'$, which is, in fact, the fraction of the total number of gaps larger than some size s. The probability density of level spacings is then given by $P(s) = -\mathrm{d}I/\mathrm{d}s$.

ILSDs for $N = F_{15} = 987$ at various values of modulation strength are presented in figure 1. From figure 1, one can see that spacing distributions which correspond to three cases ($\lambda < 2$, $\lambda = 2$ and $\lambda > 2$) seem to have three different features. $I(s)$ for $\lambda = 2$ exhibits a marginal behaviour. By measuring the slope in figure 2(a), we find that the ILSD for $\lambda = 2$ is described by $I_C(s) \sim s^{-\beta}$, with $\beta = 1/2$, which agrees quite well with the result obtained using the periodic boundary conditions [10, 12]. From figure 2(a), one can also see that there exists a cutoff of the scaling region. The behaviour of such a cutoff, which ensures $I(0) = 1$, was clearly stated in [12]. In the following, we focus attention on the shape analysis of $I(s)$ in the cases $\lambda < 2$ and $\lambda > 2$.

An electron in the Harper model for $\lambda < 2$ behaves in the same way as it does in a periodic chain [8, 9, 14–16]. Thus one may expect that the LSD follows the same law. First, let us consider a periodic chain with number of sites N. The Hamiltonian parameters are $v = 1$ and $u_n = 0$. From the matrix theory, eigenenergies of the $N \times N$ tridiagonal matrix are given by

$$E_k = 2\cos\frac{k\pi}{N+1} \qquad k = 1,\ 2,\ 3,\ \ldots,\ N. \tag{1}$$

The above equation gives the gap width $\Delta_k = |(E_{k+1} - E_k)/(4/N)|$ of adjacent levels

$$\Delta_k = N \sin\left(\frac{k+1/2}{N+1}\pi\right) \sin\left(\frac{\pi/2}{N+2}\right) \qquad k = 1,\ 2,\ 3,\ \ldots,\ N-1. \tag{2}$$

Using (2), one can exactly calculate $I(s)$. For larger-sized systems, i.e. $N \gg 1$, it can be shown that $I(s) \approx I_E(s)$ with

$$I_E(s) = (2/\pi)\ \cos^{-1}(2s/\pi). \tag{3}$$

Thus the LSD $P(s)$ is given by

$$P_E(s) = (2/\pi)^2 \sqrt{1 - (2s/\pi)^2}. \tag{4}$$

In the limit $s \to 0$, $P_E(s) \to (2/\pi)^2$ indicates level clustering. From (1) and (2), one can see that $s = 0$ is located at the edges of the spectrum of a periodic chain, where the Van Hove singularity occurs. This suggests that such a level clustering is linked to the Van Hove singularity.

Now we turn to the Harper model with $\lambda < 2$. It is clear from figure 2(b) that $\cos(\pi I/2) \approx 2As/\pi$, which yields $I(s) \approx I_E(As)$, where A are slopes of the lines in

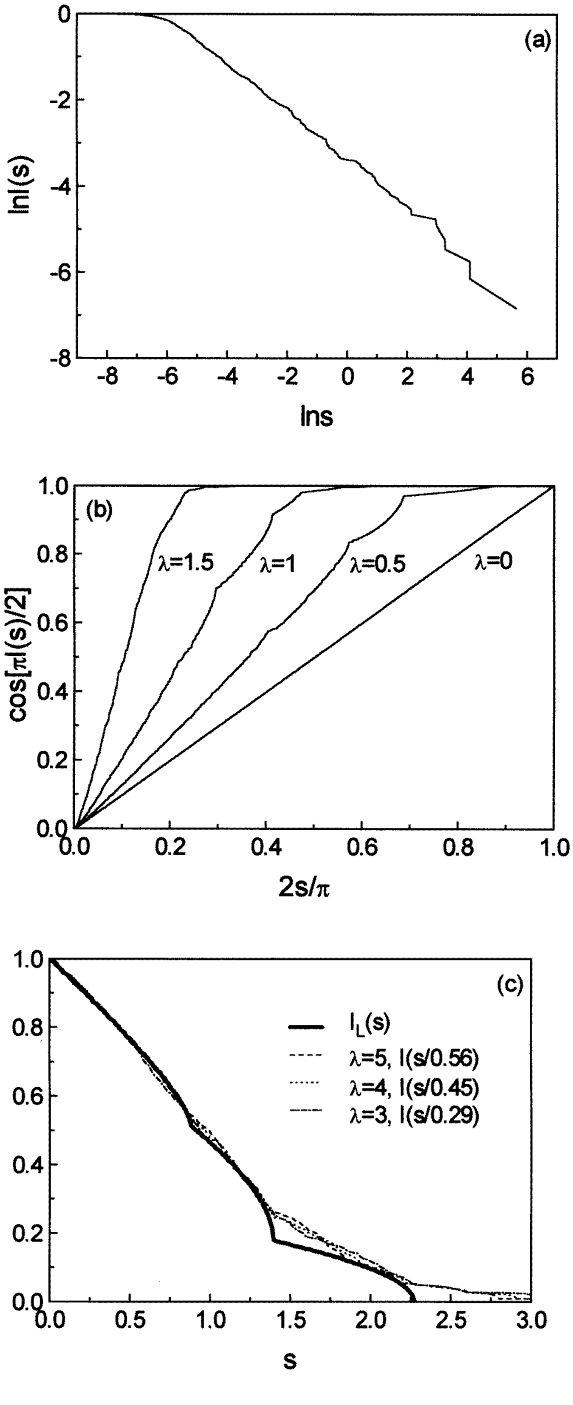

Figure 2. ILSDs $I(s)$ for the Harper model with $N = 987$: (a) a plot of $\ln I(s)$ against $\ln s$ for $\lambda = 2$ shows $I(s) \sim s^{-\beta}$, with $\beta = 1/2$; (b) a plot of $\cos(\pi I/2)$ against $2s/\pi$ for $\lambda < 2$, in which $\lambda = 0$ corresponds to the case of a periodic chain; (c) a plot of $I(s/B)$ for $\lambda = 3$, 4 and 5 with $B = 0.29$, 0.45 and 0.56, respectively, shows the fit $I(s) \approx I_L(Bs)$.

figure 2(b). Note that $\lambda = 0$ in figure 2(b) confirms the analytical result in (3) with the slope $A = 1$. Therefore the LSD of the Harper model in the metallic regime is described by $AP_E(As)$, which exhibits the same properties as the LSD $P_E(s)$ does.

For a random matrix with zero off-diagonal elements, there is a completely random sequence of eigenenergies which is given by the random site energies (this corresponds to the case of the Anderson model with strong disorder). The Poisson law describes the spacing distribution of such random sequences of eigenenergies. The surprising discovery is that, due to the localization of eigenstates, the Poisson law is suitable for describing the LSD in the whole insulating regime for the Anderson Hamiltonian [4–6]. In a quasiperiodic system, site energies in the Hamiltonian are deterministic. Consider an $N \times N$ quasiperiodic matrix with zero off-diagonal elements and diagonal part u. Eigenenergies of this matrix are given by $u_i = \lambda \cos(2\pi\sigma i)$, with $i = 1, 2, 3, \ldots, N$, which corresponds to the case of the Harper model with strong modulation $\lambda/v \gg 1$. Let us use $P_L(s)$ and $I_L(s)$ to denote the LSD and ILSD of the energies u_i, respectively. It is easy to show that $P_L(s)$ is far from the description of the Poisson law, which implies the failure of the Poisson law found by Machida and Fujita for $\lambda > 2$. In fact the fit presented in [10] is suitable only in a small range of spacing. Considering the fact that random Hamiltonians have the same spacing distribution no matter whether the off-diagonal elements are zero or not in the case of localized states, the LSD for the Harper model with $\lambda > 2$ may show the same behaviour as $P_L(s)$ does.

As expected, our calculation shows the $I(s)$ for $\lambda > 2$ can be described by $I_L(Bs)$, where B is a constant related to λ. When $\lambda \to \infty$, we have $B \to 1$. Such a fit is well demonstrated in figure 2(c). Thus, the associated $P(s)$ is given by $P(s) \approx BP_L(Bs)$. Calculation of $P_L(s)$ for large N up to $N = F_{23} = 46\,368$ shows that $P_L(s) \to 0.45$ when $s \to 0$. Therefore the LSD for the Harper model with $\lambda > 2$ has $P(s) \neq 0$ when $s \to 0$, which indicates the occurrence of level clustering. The above analysis shows that such level clustering is related to not only the localization of the eigenstates but also the spacing distribution of the site energies of the Hamiltonian.

In conclusion, we have shown that the technique of level statistics can be used to demonstrate the metal–insulator transition in quasiperiodic systems. For the Harper model, there are three possible distributions $AP_E(As)$, $P_C(s)$ and $BP_L(Bs)$ for $\lambda < 2$, $\lambda = 2$ and $\lambda > 2$, respectively, where A and B are constants related to the modulation strength λ. $P_E(s)$ is determined by the eigenenergy of a periodic chain while $P_L(s)$ is given by the energy levels which is equal to the quasiperiodic site energies. $P_C(s)$ displays an inverse power law $P_C(s) \sim s^{-\beta}$, with $\beta = 3/2$. As λ increases, the metal–insulator transition is accompanied by a crossover of the level-spacing distribution from $AP_E(As)$ to $BP_L(Bs)$ across $P_C(s)$. In the limit $s \to 0$, $P_E(As) \neq P_C(s) \to \infty$ and $P_L(Bs) \neq 0$ indicates that the level clustering occurs for three different kinds of spectrum of the studied model. It is interesting to further study the relation between the level clustering found here and the level clustering associated with the Poisson law in a disordered system. Finally, since the studied model is relevant to the kicked Harper model [17], we would like to point out that the results presented here can be applied to the study of such a quantum system with chaotic behaviour in the classical limit. In addition, the technique of level statistics was recently used to investigate the electronic properties of quasicrystals [18]. Our research may also shed light on this problem.

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